The Derivative of Brownian Motion is White Gaussian Noise

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This article derives the relation between white Gaussian noise and Brownian motion. This is very important the Kolmogorov backwards/forwards equation and in general Itô calculus.

First the definition of white noise.

Definition 0.1. (White noise) White noise is a random variable w_t such that $P[w_t|w_\tau] = P[w_t]\,\forall t > \tau \in T$.

This implies the past history contains no information on the future. We then have White Gaussian Noise is a White Noise process that is also normally distributed.

It is called "white Gaussian" because, consider a Gaussian process with the following correlation function:

$$\gamma^\rho(t+\tau,t) := E[w_{t+\tau}w_t] := \sigma^2(\rho/2)\,e^{-\sigma|\tau|} \xrightarrow[\rho\to\infty]{} \sigma^2\delta(\tau) \quad (1)$$

Then the power spectral density is:

$$f(\omega) = \int_{-\infty}^{\infty}\gamma(t+\tau,t)\,e^{-i\omega\tau}d\tau \Rightarrow f^\rho(\omega) = \frac{\sigma^2}{1+\left(\frac{\omega}{\sigma}\right)^2} \xrightarrow[\rho\to\infty]{} \sigma^2 = \text{const.} \quad (2)$$

Now let $x_t \sim N(0,\sigma^2 t)$ be a Brownian process. Then we have:

$$C_{x_t x_t}(t,\tau) = E[x_t x_\tau] = \sigma^2\min(t,\tau) = \sigma^2\begin{cases}\tau & \text{if } \tau < t\\ t & \text{if } \tau > t\end{cases} \quad (3)$$

Similarly, we have:

$$C_{\dot{x}_t\dot{x}_t}(t,\tau) = \frac{\partial}{\partial\tau}\frac{\partial}{\partial t}C_{x_t x_t}(t,\tau) = \sigma^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial t}\begin{cases}\tau & \text{if } \tau < t\\ t & \text{if } \tau > t\end{cases} \sigma^2\frac{\partial}{\partial\tau}\begin{cases}0 & \text{if } \tau < t\\ 1 & \text{if } \tau > t\end{cases} \sigma^2\delta(t-\tau) \quad (4)$$

And so we have:

$$C_{w_t w_t}(t+\tau,t) = E[w_{t+\tau}w_t] = \sigma^2\delta(\tau) = C_{\dot{x}_t\dot{x}_t}(t+\tau,t) \quad (5)$$

Thus we have that the derivative of the covariance matrix for Brownian motion is the same as the covariance matrix for white Gaussian noise, and so the derivative of Brownian motion is white Gaussian noise:

$$w_t = \frac{d\beta_t}{dt} \quad (6)$$

Note though that Brownian motion is well-defined, but white Gaussian noise is ill-defined, so we will prefer to work with the Brownian motion random variable as opposed to the white Gaussian noise random variable:

$$d\beta_t = w_t dt \quad (7)$$

This is convenient because we can just add white Gaussian noise, which is ill-defined, to a dynamic system and then change it to Brownian motion, which is well-defined.