A Friendly Introduction to Fourier Analysis

Michael Hauser

This short article gives an intuitive way to understand the Fourier transform. I've found it difficult to find this type of explanation of what the Fourier transform is; thus the motivation for this short guide. For example, very few signal processing or quantum mechanics textbooks explain the difference between the Hamel and Schauder basis, which is incredibly important for understanding Fourier analysis.

Given a signal $x(t)$, which is a map $x : \mathbb{R} \to \mathbb{R}$, the goal is to understand its frequency contents, i.e. construct a function $\hat{x}(f)$ that is a function of frequency and produces an amplitude at that frequency.

1 Fourier Series

If the signal is periodic over $\left[\frac{-T}{2}, \frac{T}{2}\right]$, i.e. $(x(t) = x(t+T))$, then one can do a Fourier series expansion with complex exponentials over this domain.

$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{i2\pi \frac{n}{T} t} \tag{1}$$

The trick here is that complex exponentials (sines and cosines) form an orthonormal (Schauder) basis over $L_2\left(\left[\frac{-T}{2}, \frac{T}{2}\right]\right)$ with the standard inner product $\langle ., . \rangle$. This means that if $e_m, e_n \in L_2\left(\left[\frac{-T}{2}, \frac{T}{2}\right]\right)$, then:

$$\langle e_m, e_n \rangle := \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} e_m^*(t)\, e_n(t)\, dt = \delta_{mn} \tag{2}$$

where δ_{mn} is 0 if $m \neq n$ and is 1 if $m = n$. In our case e_n is a basis function map $e_n : \left[\frac{-T}{2}, \frac{T}{2}\right] \to \mathbb{C}$ defined:

$$e_n(t) := e^{i2\pi \frac{n}{T} t} \tag{3}$$

Explicitly, multiplying both sides of Equation 1 by $e^{-i2\pi \frac{m}{T} t}$ and integrating over the domain yields:

$$\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t')\, e^{-i2\pi \frac{m}{T} t'} dt' = \sum_{n=-\infty}^{\infty} A_n \int_{\frac{-T}{2}}^{\frac{T}{2}} e^{i2\pi \frac{n-m}{T} t'} dt' \tag{4}$$

The integral on the right hand side is equal to T if $m = n$ and 0 if $m \neq n$. This equation reduces to the following:

$$A_m = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t')\, e^{-i2\pi \frac{m}{T} t'} dt' \tag{5}$$

These are the coefficients to the expansion of the signal in the complex exponential basis from Equation 1.

2 Fourier Transform

Combining Equations 1 and 5 yields:

$$x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t')\, e^{-i2\pi \frac{n}{T} t'} dt' \right) e^{i2\pi \frac{n}{T} t} \tag{6}$$

This is very important. Define $f_n := \frac{n}{T}$, then $\Delta f_n := f_{n+1} - f_n = \frac{1}{T}$, and we have that $\Delta f_n \to 0$ as $T \to \infty$. This then leads the coefficients/components of the vector to be rewritten as $A_n \equiv A(f_n) = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t')\, e^{-i2\pi f_n t'} dt'$. We rewrite Equation 6 in terms of our defined frequency variables:

$$x(t) = \sum_{n=-\infty}^{\infty} \left(\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t')\, e^{-i2\pi f_n t'} dt' \right) e^{i2\pi f_n t} \Delta f_n \tag{7}$$

So here, roughly speaking, $f_n := \frac{n}{T} \to f$ as $T \to \infty$ and $\Delta f_n := \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T} \to df$ as $T \to \infty$.

These vagaries are made more clear in Section 3 where it is formulated in terms of taking the closure of the countable basis in a Hilbert space to form an uncountable basis for the Hilbert space.

Remark. x is T-periodic, but if $T \to \infty$ then x is no longer periodic over $(-\infty, \infty)$. However, we are still expanding x with periodic basis functions $e^{i2\pi ft}$ which is not very intuitive. This is one of the many theoretical reasons for using wavelets as a different kind of basis function.

Remark. $f_n := \frac{n}{T}$ is a countable index and f is an uncountable index, so as we go to the integral we must take the closure over the basis vectors. More specifically, the functions $e_n : t \mapsto e^{-i2\pi \frac{n}{T} t}$ are indexed by the countable index set $n \in \mathbb{Z}$, but we need to pass it to an uncountable index set $f \in \mathbb{R}$, so we take the closure of the set of basis vectors $\{e_n\}_{n\in\mathbb{Z}}$ after passing to the limit $T \to \infty$, which gives us the uncountable set of basis vectors $\{e_f\}_{f\in\mathbb{R}}$ where $e_f : t \mapsto e^{-i2\pi ft}$.

$$x(t) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t') e^{-i2\pi ft'} dt' \right) e^{i2\pi ft} df \quad (8)$$

To explicitly relate it to the Fourier series, we write $A(f) := \int_{-\infty}^{\infty} x(t') e^{-i2\pi ft'} dt'$ from Equation 5, but of course this is the Fourier transform so it gets the following symbol:

$$\hat{x}(f) := \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt \quad (9)$$

To make an explicit comparison, in the countable basis case we have:

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n] e^{i2\pi \frac{n}{T} t} \quad (10)$$

Where we wrote $\hat{x}[n] := A_n$. This is in comparison to the uncountable basis case:

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f) e^{i2\pi ft} df \quad (11)$$

This leads to the interpretation that the Fourier transform of a signal are the coefficients to the expansion of the signal in the uncountable basis $\{e_f\}_{f\in\mathbb{R}}$ defined by $e_f : t \to e_f(t) := e^{i2\pi ft}$.

3 Notes on Hilbert spaces

Equation 3 allows us to write Equation 1 as:

$$x(t) = \sum_{n=-\infty}^{\infty} A_n e_n(t) \quad (12)$$

Now remember $(\mathcal{H}, +_{\mathcal{H}}, \cdot_{\mathcal{H}})$ is a vector space (in our case the set $\mathcal{H} = L_2\left(\left[\frac{-T}{2}, \frac{T}{2}\right]\right)$), where vector addition and scalar multiplication are defined pointwise (for $x, y \in \mathcal{H}$ then $(x +_{\mathcal{H}} y)(t) := x(t) +_{\mathbb{C}} y(t)$, and for $\lambda \in \mathbb{C}$ as the scalar field, then $(\lambda \cdot_{\mathcal{H}} x)(t) := \lambda \cdot_{\mathbb{C}} x(t)$). In this way, functions are, roughly speaking, infinite dimensional vectors. The $\mathbb{C}$-vector space $(\mathcal{H}, +_{\mathcal{H}}, \cdot_{\mathcal{H}})$ with the (sesquilinear) inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ together as the tuple $(\mathcal{H}, +_{\mathcal{H}}, \cdot_{\mathcal{H}}, \langle ., . \rangle)$ is called a Hilbert space.

An enormously important property of the Hilbert space is that the inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ induces a norm $|| \cdot || : \mathcal{H} \to \mathbb{R}$ defined by $||x|| := \sqrt{\langle x, x \rangle}$, and the Hilbert space is *complete* with respect to this norm (i.e. every Cauchy sequence converges). This is a very important distinction between the Hamel basis (finite) and Schauder basis (infinite). The Hamel basis can *only* represent a vector as a finite sum (as in the case in the standard linear algebra course). In comparison a Schauder basis can both represent vectors as a finite sum (e.g. the signal is a finite sum of sine waves), as well as an infinite sum and converge to the vector in the limit. In this way the Schauder basis has much greater expressive power.

Remark. Note that $e_m \in L_2(\mathbb{R})$ is a *function*. $e_m(t)$ is the value of the function e_m at the point $t \in \mathbb{R}$, i.e. $e_m(t)$ is a *point, not a function*. The *function* is e_m and is defined by how it maps a point $t \in \mathbb{R}$ to a new point $e_m(t)$, so in our case we can write $e_m : t \mapsto e_m(t) := e^{-i2\pi \frac{m}{T} t}$. This is a subtle point, but very often misunderstood by people in signal processing. So for example in Equation 2, you can take the inner product of two functions/vectors $e_m, e_n \in L_2(\mathbb{R})$ as $\langle e_m, e_n \rangle$, but to take the inner product of two points $\langle e_m(t), e_n(t) \rangle$ is nonsense (the inner product is a map $\langle \cdot, \cdot \rangle : L_2(\mathbb{R}) \times L_2(\mathbb{R}) \to \mathbb{R}$, and $e_m, e_n \in L_2(\mathbb{R})$, whereas $e_m(t), e_n(t) \notin L_2(\mathbb{R})$. It would be more clear to write $\langle \cdot, \cdot \rangle_{L_2}$ instead of $\langle \cdot, \cdot \rangle$, but it is also just too much notation).

To understand Fourier analysis in the context of (separable) Hilbert spaces, we should really write Equation 12 as:

$$x = \sum_{n=-\infty}^{\infty} \langle e_n, x \rangle \, e_n \tag{13}$$

where $A_n := \langle e_n, x \rangle$ (it is a separable Hilbert space because it has a Shauder basis $\{e_n\}_{n \in \mathbb{Z}}$ that is (i.) countable and (ii.) orthonormal). In quantum mechanics this is sometimes called inserting the resolution of identity. The only difference between this and a basis expansion of a vector in a normal linear algebra course is that this is an infinite dimensional (Schauder) basis, whereas a standard linear algebra course mostly deals with finite dimensional (Hamel) basis. In the uncountably infinite basis case, the sum in Equation 13 is an integral:

$$x = \int_{-\infty}^{\infty} \langle e_f, x \rangle \, e_f df \tag{14}$$

The difference between Equations 13 and 14 is that Equation 13 expands the signal in a countable basis $e_n \in L_2\left(\left[\frac{-T}{2}, \frac{T}{2}\right]\right)$ such that $e_n : t \mapsto^{-i2\pi \frac{n}{T} t}$, whereas Equation 14 expands the signal in an uncountable basis $e_f \in L_2(\mathbb{R})$ such that $e_f : t \mapsto^{-i2\pi f t}$ for $f \in \mathbb{R}$, but otherwise they are intuitively the same thing.

4 As an operator

The discrete Fourier series is given by:

$$\hat{x}[n] := \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t') \, e^{-i2\pi \frac{n}{T} t'} dt' \tag{15}$$

whereas the Fourier transform is given by:

$$\hat{x}(f) := \int_{-\infty}^{\infty} x(t') \, e^{-i2\pi f t'} dt' \tag{16}$$

The Fourier transform can be understood as an operator as well. The Fourier operator $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ (it actually maps between Schwartz spaces), where we have written $\hat{x} := \mathcal{F}(x)$. As an operator, it takes functions and maps them to functions. Remember x is a function and $x(t)$ is a point, so the Fourier operator $\mathcal{F}$ acts on the function x, and is defined by the value it produces at each point f.

$$\begin{aligned} \mathcal{F} : x \mapsto & \hat{x} := \mathcal{F}(x) \\ \hat{x}(f) := \mathcal{F}(x)(f) := & \int_{-\infty}^{\infty} x(t') \, e^{-i2\pi f t'} dt' \end{aligned} \tag{17}$$

In this way the Fourier operator is just an operator that acts on a function and produces a new function, where the new function is defined pointwise for $f \in \mathbb{R}$ (for the Fourier transform), or $n \in \mathbb{Z}$ (for the Fourier series).