Derivation of Brownian Motion from the Random Walk

Michael Hauser

This article gives a short derivation of Brownian motion from the random walk in an easy to follow way. First define the step random variable Z_k, where we take a step to the right with probability p and we take a step to the left with probability $1-p$:

$$Z_k := \begin{cases} +\Delta x & \text{with probability } p \\ -x & \text{with probability } q = 1-p \end{cases} \tag{1}$$

Then the total distance traveled is the sum:

$$S_n := \sum_{k=1}^{n} Z_k \tag{2}$$

To find the mean and variance of S_n, use the characteristic function, where $E[\cdot]$ is the expectation:

$$\varphi_{S_n}(u) := E\left[e^{iS_n u}\right] = E\left[e^{i\sum_{k=1}^{n} Z_k u}\right] = E\left[\prod_{k=1}^{n} e^{iZ_k u}\right] = \prod_{k=1}^{n} E\left[e^{iZ_k u}\right] = E\left[e^{iZ_k u}\right]^n = \left(pe^{i\Delta xu} + qe^{-i\Delta xu}\right)^n \tag{3}$$

Let there be a total of $n = \frac{t}{\Delta t}$ steps over time t and time-step Δt, then:

$$\varphi_{S_n}(u) = \left(pe^{i\Delta xu} + qe^{-i\Delta xu}\right)^{t/\Delta t} \tag{4}$$

Remark. (Moment Generating Functions) $\varphi_x(k) = E\left[e^{ikx}\right] \Rightarrow \frac{d^m\varphi_x}{dk^m}(k) = \left(\frac{d}{dk}\right)^m E\left[e^{ikx}\right] = E\left[(ix)^m e^{ikx}\right]$. This means that at $k=0$ we have: $\frac{d^m\varphi_x}{dk^m}(k=0) = i^m E[x^m]$.

From the remark, we have the result:

$$E[x^m] = \frac{1}{i^m}\frac{d^m\varphi_x}{dk^m}(0)\,\forall m \in \mathbb{N} \tag{5}$$

With this, and letting $p = q = 1/2$:

$$E\left[(S_n)^m\right] = \frac{1}{i^m}\frac{d^m\varphi_{S_n}}{du^m}(u=0)\,\forall m \in \mathbb{N} \tag{6}$$

For $m = 1$ (i.e. the mean):

$$E[S_n] = \frac{1}{i}\frac{d\varphi_{S_n}}{du}(0) = \frac{1}{i}\left(\frac{1}{2}\right)^{t/\Delta t} \xrightarrow[\Delta t \to 0]{} 0 \tag{7}$$

For $m = 2$ (i.e. the variance):

$$E\left[(S_n)^2\right] = \frac{1}{i^2}\frac{d^2\varphi_{S_n}}{du^2}(0) = \text{...lots of algebra...} = t\frac{\Delta x^2}{\Delta t} \xrightarrow[\Delta x, \Delta t \to 0]{} t\sigma^2 \tag{8}$$

Thus we have, with the limiting <u>definition</u> (an example of Einstein's genius) $\frac{\Delta x^2}{\Delta t} \xrightarrow[\Delta x, \Delta t \to 0]{} \sigma^2$. Using these results, we substitute this limiting definition into Equation 4 (along with $p = q = 1/2$) to generate the density function in the continuous limit:

$$\varphi_{S_n}(u) = \left(\frac{1}{2}\left(e^{i\Delta xu} + e^{-i\Delta xu}\right)\right)^{t/\Delta t} \xrightarrow[\Delta x, \Delta t \to 0]{} \varphi_{x_t}(u) = \left(\frac{1}{2}\left(e^{iu\sigma\sqrt{\Delta t}} + e^{-iu\sigma\sqrt{\Delta t}}\right)\right)^{t/\Delta t} = \cos\left(u\sigma\sqrt{\Delta t}\right)^{t/\Delta t} \tag{9}$$

Note that the name of the *discrete* stochastic variable is S_n, whereas after the limit we have the *continuous* stochastic variable x_t.

Now do two Taylor expansions and ignore the higher order terms. The first Taylor expansion:

$$\varphi_{x_t}(u) = \cos\left(u\sigma\sqrt{\Delta t}\right)^{t/\Delta t} = \left(1 - \frac{1}{2}\left(u\sigma\sqrt{\Delta t}\right)^2 + \mathcal{O}\left(\sqrt{\Delta t}^3\right)\right)^{t/\Delta t} \quad (10)$$

The second Taylor expansion:

$$\log \varphi_{x_t}(u) = \log\left(1 - \frac{1}{2}u^2\sigma^2\Delta t\right)^{t/\Delta t} = \frac{t}{\Delta t}\log\left(1 - \frac{1}{2}u^2\sigma^2\Delta t\right) = \frac{t}{\Delta t}\left(-\frac{1}{2}u^2\sigma^2\Delta t + \mathcal{O}\left(\Delta t\right)^2\right) \quad (11)$$

We can finally rearrange this final term as follows:

$$\log \varphi_{x_t}(u) = -\frac{1}{2}u^2\sigma^2 t \iff \varphi_{x_t}(u) = e^{-\frac{1}{2}u^2\sigma^2 t} \iff x_t \sim N\left(0, \sigma^2 t\right) \quad (12)$$

Conclusion: We have our result, the discrete random walk turns into Brownian motion in the continuous limit.